Tricritical point of the J_1-J_2 Ising model on a hyperbolic lattice

R. Krcmar,[1] T. Iharagi,[2] A. Gendiar,[1,3] and T. Nishino[2]

[1]*Institute of Electrical Engineering, Centre of Excellence CENG, Slovak Academy of Sciences, Dúbravská cesta 9, SK-841 04, Bratislava, Slovakia*

[2]*Department of Physics, Graduate School of Science, Kobe University, Kobe 657-8501, Japan*

[3]*Institute for Theoretical Physics C, RWTH Aachen University, D-52056 Aachen, Germany*

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The ferromagnetic-paramagnetic phase transition of the two-dimensional frustrated Ising model on a hyperbolic lattice is investigated by use of the corner transfer matrix renormalization group method. The model contains a ferromagnetic nearest-neighbor interaction J_1 and a competing antiferromagnetic interaction J_2. A mean-field-like second-order phase transition is observed when the ratio $\kappa = J_2/J_1$ is less than 0.203. In the region $0.203 < \kappa < 1/4$, the spontaneous magnetization is discontinuous at the transition temperature. Such tricritical behavior suggests that the phase transitions on hyperbolic lattices need not always be mean-field-like.

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I. INTRODUCTION

The Ising model on the Cayley tree is known for its singular property, where the magnetic susceptibility of the spin at the root of the tree diverges at a temperature T_c despite the fact that there is no singularity in the partition function of the whole system [1]. This is a kind of phase transition which can be explained by the Ising model on the Bethe lattice. It is known that the Ising model on hyperbolic lattices, which are negatively curved in the two-dimensional (2D) space [2], exhibits similar aspects [3–5]. The universality class of the ferromagnetic-paramagnetic phase transition of this model has been so far considered to be mean-field-like. Recent numerical studies have supported this conjecture [6–9].

In this paper we study the effects of the antiferromagnetic next-nearest-neighbor (NNN) interaction J_2, which competes with the ferromagnetic nearest-neighbor (NN) interaction J_1, on the ferromagnetic-paramagnetic phase transition of the 2D Ising model on a hyperbolic lattice. We use the corner transfer matrix renormalization group (CTMRG) method [13–15], which is a variant of the density matrix renormalization group (DMRG) method [16–19], for calculations of thermodynamic functions. As we show in the following, the transition temperature T_0 monotonically decreases with increasing frustration parameter $\kappa = J_2/J_1$ in the region $0 \leqslant \kappa < 1/4$, where the ground-state spin configuration is completely ferromagnetic. We find that there is a tricritical point when the parameter κ is equal to $\kappa_c = 0.203$. The ferromagnetic-paramagnetic phase transition is of the second order for $0 \leqslant \kappa \leqslant \kappa_c$, whereas it turns into a first-order one for $\kappa_c < \kappa < 1/4$.

In the next section, we explain the so-called (5,4) lattice in the 2D hyperbolic space and introduce the Ising Hamiltonian on it. As a theoretical ideal, we consider a phase transition on a Bethe lattice with coordination number 4, which is equivalent to the $(\infty,4)$ hyperbolic lattice. In Sec. III we present numerical results. The temperature dependence of the free energy and spontaneous and induced magnetizations is shown. We analyze these thermodynamic functions around the transition temperature T_0 for several values of κ, and determine the critical exponents α, β, and δ. We summarize the observed phase transition in the last section.

II. FRUSTRATED ISING MODEL ON A HYPERBOLIC LATTICE

We consider the hyperbolic 2D lattice shown in Fig. 1, where four pentagons share their apexes. Such a lattice is conventionally called the (5,4) lattice, where the number 5 represents the number of sides of each pentagon and the number 4 is the coordination number. Consider the Ising model on this lattice, where on each lattice site labeled by i there is an Ising spin variable $\sigma_i = \pm 1$. We assume ferromagnetic interactions between NN spin pairs, shown by the full lines in Fig. 1, and antiferromagnetic interactions between NNN pairs, shown by the dashed lines. The Hamiltonian of the system is represented as

$$\mathcal{H} = -J_1 \sum_{\langle ij \rangle = \mathrm{NN}} \sigma_i \sigma_j + J_2 \sum_{\langle ik \rangle = \mathrm{NNN}} \sigma_i \sigma_k, \tag{1}$$

where $J_1 > 0$ is the ferromagnetic coupling constant between the NN pairs $\langle ij \rangle$ and $J_2 > 0$ is the antiferromagnetic one

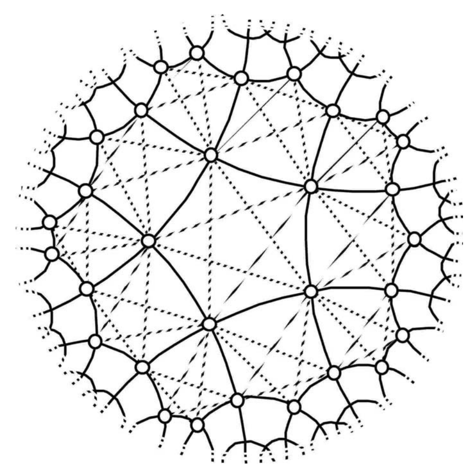

FIG. 1. (5,4) hyperbolic lattice drawn in the Poincaré disk. The open circles represent the Ising spin sites. The next-nearest-neighbor interactions are here represented by the dashed lines inside the pentagons.

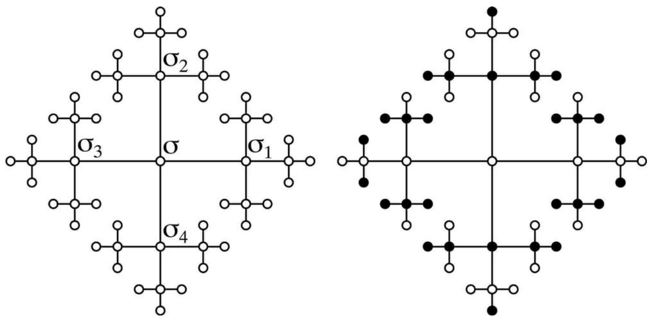

FIG. 2. Ground-state spin configurations for $\kappa<1/4$ (left) and $\kappa>1/4$ (right) on the $(\infty,4)$ lattice, which coincides with the Bethe lattice with the coordination number 4. Note that only a finite number of spins is here depicted from the $(\infty,4)$ lattice. Open circles represent spin variable $\sigma=+1$ whereas the full circles correspond to $\sigma=-1$.

between the NNN pairs $\langle ik\rangle$. Let us define a parameter $\kappa = J_2/J_1$ that represents the strength of the frustration.

For the purpose of obtaining brief insight into the phase structure of the (5,4) Ising model, we observe the model in a wider framework. Let us introduce the Ising model on the $(n,4)$ lattice where four n-gons (polygons of the nth order) meet at each lattice point. When the number of sides n ($\geqslant 5$) is a multiple of 4, including the case $n=\infty$, the ground-state spin configuration at zero temperature is easily obtained. Figure 2 shows the ground-state configurations for the case $n=\infty$, where the lattice is nothing but the Bethe lattice with the coordination number 4. For the complete ferromagnetic configuration shown on the left, the energy expectation value per site is

$$\varepsilon_{\text{ferro}} = -2J_1 + 4J_2, \tag{2}$$

and for the "up-up-down-down" structure shown on the right, the value is

$$\varepsilon_{\text{uudd}} = -4J_2. \tag{3}$$

Therefore, the energy crossover $\varepsilon_{\text{ferro}}=\varepsilon_{\text{uudd}}$ is located at $J_1 = 4J_2$, or equivalently at $\kappa=1/4$. This ground-state alternation is common for all the cases where n ($\geqslant 5$) is a multiple of 4. If not, the ground-state spin configuration for large κ is not unique and is probably disordered. In the case of the (5,4) lattice, one of the ground states in the large-κ region can be constructed by joining the pentagons with either "up-up-up-down-down" or "up-up-down-down-down" spin configurations. After some algebra, one obtains the energy per site

$$\varepsilon_{\text{uuudd}} = \varepsilon_{\text{uuddd}} = -\frac{2}{5}J_1 - \frac{12}{5}J_2 \tag{4}$$

for the assumed configurations. Hence the energy crossover $\varepsilon_{\text{ferro}}=\varepsilon_{\text{uuudd}}$ also occurs at $J_1=4J_2$.

At finite temperature, a ferromagnetic-paramagnetic phase transition is observed in the small-κ region [20]. Consider a single-site mean-field approximation on an arbitrary $(n,4)$ lattice. The mean-field variable h is expressed as

$$h = (-4J_1 + 8J_2)\langle\sigma\rangle = -(4-8\kappa)J_1\langle\sigma\rangle, \tag{5}$$

where $\langle\sigma\rangle$ is the expectation value of the Ising spin. The self-consistent condition for $\langle\sigma\rangle$ leads to a ferromagnetic-paramagnetic phase transition with the critical temperature $T_c^{\text{MF}}(\kappa)=(4-8\kappa)J_1/k_B$, where k_B is the Boltzmann constant. Within this approximation, the transition is always of the second order in the region $0\leqslant\kappa<1/4$, since the effect of J_2 appears as a rescaling of the mean-field variable h as in Eq. (5). It should be noted that $T_c^{\text{M.F.}}(\kappa=1/4)=2J_1/k_B$ is larger than zero. The mean-field approximation predicts another ordered state in the region $\kappa>1/4$, where the up-up-down-down spin configuration is favored if the lattice geometry allows the ordering.

An improvement to the mean-field approximation is achieved by increasing the number of sites that are not averaged. The simplest case is the Bethe approximation, which treats additional spins σ_1, σ_2, σ_3, and σ_4 that surround the central site σ, as shown in Fig. 2 (left). On an arbitrary $(n,4)$ lattice, the mean field for the surrounding four spins σ_1, σ_2, σ_3, and σ_4 is given by

$$h_a = (-3J_1 + 6J_2)\langle\sigma\rangle. \tag{6}$$

As an effect of the next-nearest-neighbor interaction, the central spin σ also experiences a mean field, but of different strength,

$$h_b = 8J_2\langle\sigma\rangle, \tag{7}$$

in addition to the direct ferromagnetic interaction with the surrounding spins $-J_1\sigma(\sigma_1+\sigma_2+\sigma_3+\sigma_4)$. Considering these interaction terms, one obtains the self-consistent relation

$$\langle\sigma\rangle = \frac{1}{Z}\sum \sigma \exp[-\beta h_a(\sigma_1+\sigma_2+\sigma_3+\sigma_4) - \beta h_b\sigma + \beta J_1(\sigma_1 + \sigma_2+\sigma_3+\sigma_4)\sigma - \beta J_2(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_4+\sigma_4\sigma_1)], \tag{8}$$

where $\beta=1/k_BT$ and where Z is the partition function

$$Z = \sum \exp[-\beta h_a(\sigma_1+\sigma_2+\sigma_3+\sigma_4) - \beta h_b\sigma + \beta J_1(\sigma_1+\sigma_2 + \sigma_3+\sigma_4)\sigma - \beta J_2(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_4+\sigma_4\sigma_1)]. \tag{9}$$

The configuration sums in Eqs. (8) and (9) are taken over the five spins σ, σ_1, σ_2, σ_3, and σ_4. The factorization

$$W(\sigma_i,\sigma) = \exp\left(-\beta h_a\sigma_i + \beta J_1\sigma_i\sigma - \beta\frac{h_b}{4}\sigma\right) \tag{10}$$

for $i=1$, 2, 3, and 4 further simplifies the expression so that the partition function has the form

$$Z = \sum W(\sigma_1,\sigma)W(\sigma_2,\sigma)W(\sigma_3,\sigma)W(\sigma_4,\sigma)\exp[-\beta J_2(\sigma_1\sigma_2 + \sigma_2\sigma_3+\sigma_3\sigma_4+\sigma_4\sigma_1)]. \tag{11}$$

Since it is not a trivial task to find an analytical solution of the self-consistent Eqs. (6)–(9), we solved them numerically. We use the parametrization $J_1=1$ and $k_B=1$ throughout this paper in the numerical calculations. Figure 3 shows the calculated spontaneous magnetization $M=\langle\sigma\rangle$. The second-order phase transition is detected in the whole region $0\leqslant\kappa$

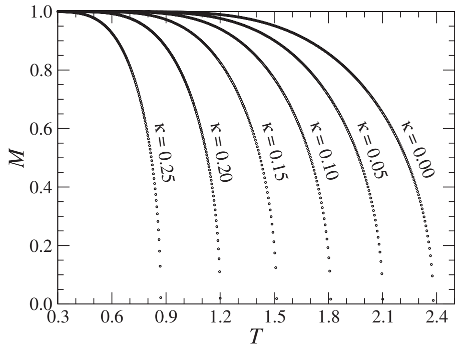

FIG. 3. Spontaneous magnetization $\langle\sigma\rangle$ obtained by the Bethe approximation.

$<1/4$. As observed in the single-site mean-field approximation, the transition is of the second order, and the transition temperature remains finite even at $\kappa=1/4$. Further improvement of the Bethe approximation can be achieved by means of a gradual increase of the unaveraged spin sites. A series of such approximations is known as the coherent anomaly method (CAM) [21]. Here, we do not proceed with the CAM analysis; we perform extensive numerical calculations by the CTMRG method instead.

It is known that the spin expectation value $\langle\sigma\rangle$ can be calculated exactly at the root of the Cayley tree, which can be treated as a Bethe lattice [1]. For the frustrated Ising model on the $(\infty,4)$ lattice shown in Fig. 2, the expectation value is expressed as

$$\langle\sigma\rangle = \frac{1}{Z'}\sum \sigma W'(\sigma_1,\sigma)W'(\sigma_2,\sigma)W'(\sigma_3,\sigma)W'(\sigma_4,\sigma) \times\exp[-\beta J_2(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_4+\sigma_4\sigma_1)] \qquad (12)$$

with the definition of the effective partition function

$$Z' = \sum W'(\sigma_1,\sigma)W'(\sigma_2,\sigma)W'(\sigma_3,\sigma)W'(\sigma_4,\sigma) \times\exp[-\beta J_2(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_4+\sigma_4\sigma_1)], \qquad (13)$$

where the new factor $W'(\sigma_i,\sigma)$ represents a Boltzmann weight for a branch that connects the root spin σ with the nearest spin site σ_i (cf. Fig. 2). This new factor $W'(\sigma_i,\sigma)$ can be calculated from $W(\sigma_i,\sigma)$ in Eq. (10) by repeating the application of the recursive transformation,

$$W_{\rm new}(\sigma_i,\sigma) = \sum_{s_1,s_2,s_3} W(s_1,\sigma)W(s_2,\sigma)W(s_3,\sigma) \times\exp[\beta J_1\sigma_i\sigma-\beta J_2(\sigma s_1+s_1s_2+s_2s_3+s_3\sigma)] \qquad (14)$$

many times until it converges [1]. Thus $W'(\sigma_i,\sigma)$ contains the effect of distant sites on the Bethe lattice. Figure 4 shows the spontaneous magnetization $M=\langle\sigma\rangle$ calculated by Eq. (12) using $W'(\sigma_i,\sigma)$ numerically obtained from Eq. (14). The transition is of the second order in the region $0\leqslant\kappa\leqslant 0.183$ and is of the first order in $0.184\leqslant\kappa<1/4$. One can carry out perturbative calculations to ensure that the transition temperature on the Bethe lattice is zero at $\kappa=1/4$. The difference between Figs. 3 and 4 comes from the effect of distant interacting spin sites, which might be essential in the tricritical behavior on the (5, 4) lattice as studied in the next section.

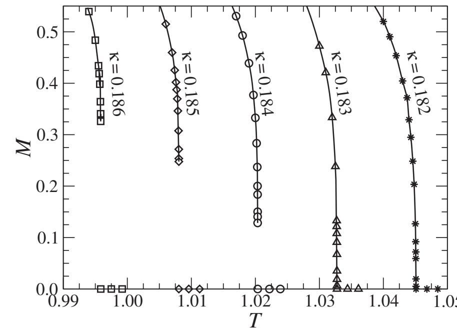

FIG. 4. Spontaneous magnetization of the J_1-J_2 Ising model on the $(\infty,4)$ lattice.

III. NUMERICAL RESULTS BY CTMRG

In this section we analyze the thermodynamic property of the Ising model on the (5, 4) lattice by use of the CTMRG method [13–15]. The method is a variant of the DMRG method [16–18] applied to 2D classical models [22]. It is known that the partition function Z of a square-shaped finite-size system can be calculated as a trace of the fourth power of the so-called corner transfer matrix (CTM), which represents the Boltzmann weight of a quadrant of the whole system [1]. Although the matrix dimension of the CTM, which is denoted by C, increases exponentially with the linear size of the system, it is possible to transform it into a renormalized one $\tilde{C}$ with a smaller matrix dimension m [23] by means of the RG transformation obtained from the diagonalization of $\rho=C^4$ or C [1,13,15]. This transformation is not exact but is highly accurate in the sense that $\tilde{Z}=\mathrm{Tr}\,\tilde{C}^4$ is a good approximation of $Z=\mathrm{Tr}\,C^4$. One can precisely calculate thermodynamic (or one-point) functions, such as the free energy $F=-k_BT\ln\tilde{Z}$ and the spontaneous magnetization M, for a sufficiently large finite-size system by use of the CTMRG method. Since the $(n,4)$ lattice can be divided into four equivalent parts (the quadrants), which share the central site σ on their edges, it is also possible to apply the CTMRG method to statistical models on these lattices [8,9,12].

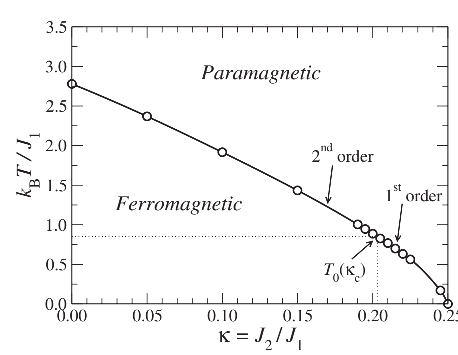

FIG. 5. Phase diagram of the J_1-J_2 Ising model on the (5, 4) hyperbolic lattice.

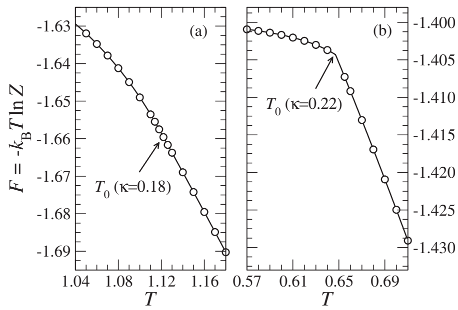

FIG. 6. Dependence of the free energy $F(\kappa;T)$ on temperature when κ=(a) 0.18 and (b) 0.22.

In order to study critical phenomena correctly on hyperbolic lattices, we put the following remarks. We always consider a lattice system whose linear size L is several times larger than the corresponding correlation length ξ, so that the central site σ is sufficiently away from the system boundary. The lattice sites in the area within the distance of the order of ξ from the system boundary are affected by the imposed ferromagnetic boundary condition, where all the Ising spins at the system boundary point to the same direction. It should also be noted that the portion of such sites that are "near the boundary" in the hyperbolic geometry remains finite even in the thermodynamic limit $L\to\infty$ [10,11], where the situation is similar to the case of the Cayley tree [1]. Disregarding all these sites near the boundary, we focus on the thermodynamic properties of the Ising spins deep inside the system [3,12].

Figure 5 shows the phase diagram of the system in the parameter region $0\leq\kappa<1/4$. The ferromagnetic-paramagnetic phase boundary is determined from the temperature dependence of the free energy $F(\kappa;T)$ and the spontaneous magnetization $M(\kappa;T)$ which we show in the following. As an effect of the competing interactions, the transition temperature $T_0(\kappa)$ monotonically decreases with increasing κ towards $T_0(1/4)=0$. In the region $0\leq\kappa\leq0.203$ the transition is of the second order. In contrast, when $0.203<\kappa<1/4$, we observe a first-order transition; the tricritical point is located at $\kappa_c=0.203$. Figure 6 shows the free energy $F(\kappa;T)$ at $\kappa=0.18$ and 0.22. In the region $0.203<\kappa<1/4$, the free energy $F(\kappa;T)$ is not a differentiable function at the transition temperature, as shown in Fig. 6(b).

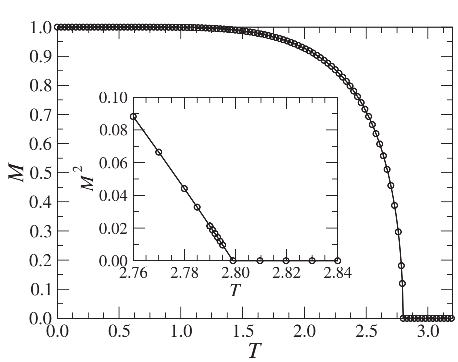

FIG. 7. Spontaneous magnetization M for $\kappa=0$.

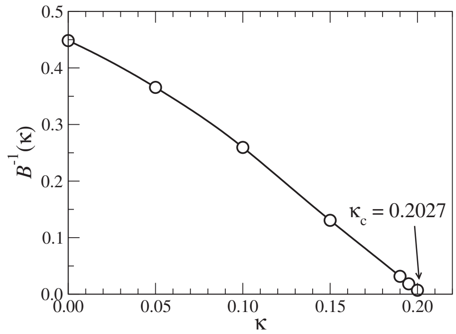

FIG. 8. Inverse of the prefactor $B(\kappa)$, which characterizes the mean-field-like transition observed in the spontaneous magnetization.

Figure 7 shows the spontaneous magnetization M and its square M^2, when there is no frustration, $\kappa=0$. The squared magnetization M^2 is proportional to T_0-T, a behavior that agrees with the critical exponent $\beta=1/2$. In the second-order transition region $0\leq\kappa<0.203$, the universality class remains mean-field-like and the magnetization curve satisfies the scaling form

$$M(\kappa;T)=B(\kappa)[T_0(\kappa)-T]^{1/2} \tag{15}$$

around the transition temperature $T_0(\kappa)$. The prefactor $B(\kappa)$ is an increasing function of κ and diverges at a certain point $\kappa=\kappa_c$. Figure 8 shows the inverse of the prefactor $B(\kappa)$ which linearly decreases to zero in the vicinity of κ_c. We obtain $\kappa_c=0.2027$ from the linear fitting.

One can also estimate κ_c out of the discontinuity in the spontaneous magnetization $M(\kappa;T)$ in the region $0.203<\kappa<1/4$. Figure 9 shows $M(\kappa;T)$ around $\kappa=\kappa_c$. We calculate the discontinuity function (or the jump in the magnetization) $D(\kappa)=M(\kappa;T_0^-)$, where T_0^- corresponds to a temperature just below the transition temperature $T_0(\kappa)$. As shown in Fig. 10, the discontinuity function satisfies the relation

$$D(\kappa)\propto(\kappa-\kappa_c)^{1/4} \tag{16}$$

around $\kappa=\kappa_c$. Performing the linear fitting shown by the dashed lines, we obtain $\kappa_c=0.2033$. Comparing this value with $\kappa_c=0.2027$ obtained from the data in the second-order region, we conclude that the tricritical point is located at $\kappa_c=0.203\pm0.001$.

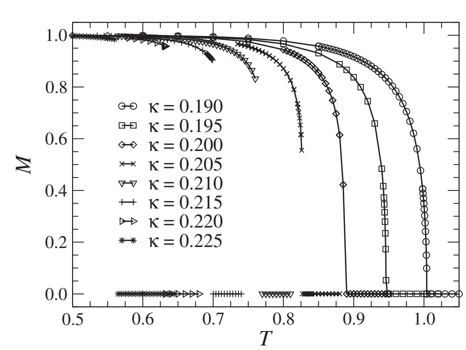

FIG. 9. Temperature dependence of the spontaneous magnetization $M(\kappa;T)$ for several values κ around κ_c.

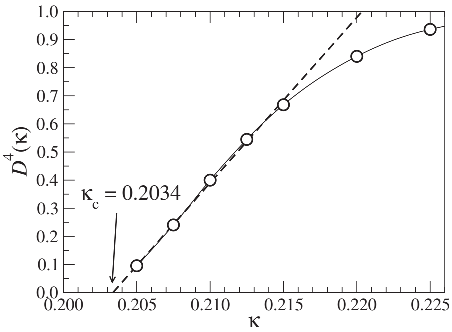

FIG. 10. Discontinuity function of the spontaneous magnetization at the transition temperature T_0 in the first-order transition region $\kappa_c < \kappa < 1/4$.

The observed tricritical behavior around $\kappa = \kappa_c$ is in accordance with the Landau free energy

$$F(M,t) = aM^6 + b(\kappa_c - \kappa)M^4 + ctM^2, \qquad (17)$$

where a, b, and c are positive constants or slowly varying functions of temperature. In the second-order transition region $\kappa < \kappa_c$, the second and the third terms in $F(M,t)$ are dominant in the vicinity of the phase transition, and the parameter t coincides with $[T - T_0(\kappa)]/T_0(\kappa)$. Neglecting the first term in $F(M,t)$ below $T < T_0(\kappa)$, we can obtain the spontaneous magnetization M that minimizes $F(M,t)$ from the equation $4b(\kappa_c - \kappa)M^2 + 2ct = 0$. The behavior $M^2 \propto |t|$ coincides with the numerical result shown in Fig. 7. In the first-order transition region $\kappa > \kappa_c$, all three terms in $F(M,t)$ are important for the minimum of the free energy. After short calculations, one can confirm that the jump of the spontaneous magnetization at the transition temperature coincides with Eq. (16), which we have verified from the numerical data shown in Fig. 10.

At the tricritical point $\kappa = \kappa_c$, the second term in $F(M,t)$ in Eq. (17) vanishes. Thus, the spontaneous magnetization is determined from $6aM^4 + 2ct = 0$, and M^4 is proportional to $T - T_0$. The dependence of M^4 with respect to temperature T obtained from the numerical calculation is shown in Fig. 11. The data are in accordance with $M \propto (T - T_0)^{1/4}$, which corresponds to the exponent $\beta = 1/4$ at tricriticality. In the same manner it is expected that the specific heat diverges as $(T_0 - T)^{-1/2}$, which corresponds to the critical exponent $\alpha = 1/2$.

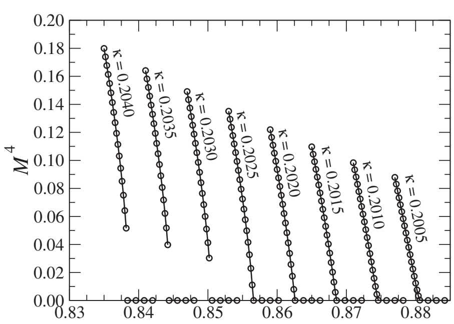

FIG. 11. Fourth power of the spontaneous magnetization around $\kappa = \kappa_c$.

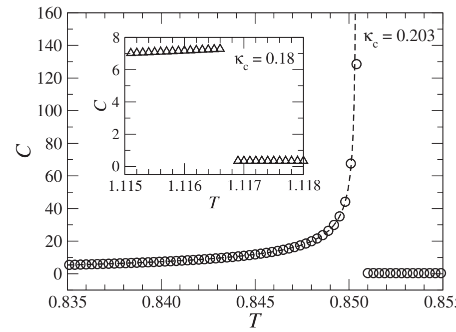

FIG. 12. Specific heat around the tricritical point.

Figure 12 shows the numerically calculated specific heat at $\kappa_c = 0.203$, which agrees with the expected temperature dependence. For comparison, in the inset of Fig. 12, we show the data at $\kappa = 0.18$, which agree with $\alpha = 0$.

Effect of an external magnetic field H may be included in the Landau free energy by adding the interaction term dHM to $F(M,t)$ in Eq. (17), where d is a positive constant. From the assumed form of the free energy it is expected that M^3 and M^5 are, respectively, proportional to $T_0 - T$ when $\kappa < \kappa_c$ and when $\kappa = \kappa_c$. For confirmation, we observe the induced magnetization at criticality when $\kappa \leq \kappa_c$. Figure 13 shows the induced M with respect to H. In the region of the second-order phase transition, we obtained the magnetic exponent $\delta = 3$ as expected. However, the value of the exponent δ is around 7 at the tricritical point, not 5 as expected from the Landau free energy $F(M,t) + dHM$. Such pathological behavior in the induced magnetization is a remaining piece of the puzzle of the current study on the (5,4) hyperbolic lattice. Considering the J_1-J_2 Ising model on the $(\infty,4)$ Bethe lattice, we obtain $\delta \sim 6$ at the tricritical point from numerical calculations. Future studies on $(n,4)$ lattices for $n \geq 6$ would provide information about these unexpected values of δ.

IV. CONCLUSIONS

We have studied a ferromagnetic-paramagnetic phase transition of the J_1-J_2 Ising model on the (5,4) hyperbolic lattice. A tricritical point has been found when the ratio $\kappa = J_2/J_1$, which represents the strength of frustration, is equal to 0.203. It should be noted that the presence of the first-order transition cannot be obtained by the single-site mean-

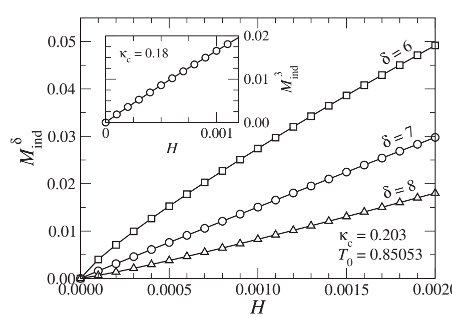

FIG. 13. Induced magnetization at the tricritical point $\kappa = \kappa_c$ and (inset) at the transition temperature when $\kappa < \kappa_c$.

field approximation applied to this system. This is in contrast to the known fact that the phase transition of the nearest-neighbor Ising, Potts, and clock models exhibits a mean-field nature [3,6–9,12].

The observed second-order phase transition in the parameter region $\kappa < \kappa_c = 0.203$ belongs to the mean-field universality class, which is characterized by the exponents $\alpha=0$, $\beta=1/2$, and $\delta=3$. At the tricritical point we observe $\alpha=1/2$, $\beta=1/4$, which are in accordance with the Landau free energy written as an even polynomial of the order parameter. The observed value of the exponent $\delta \approx 7$ at the tricritical point is the only exception, and requires further detailed studies.

As an effect of the frustration, the entropy of the ordered phase will be enhanced compared with the ordered state that has the same spontaneous magnetization under $J_2=0$. We conjecture that this enhancement effect creates a minimum in the Landau free energy, which may be the reason for the first-order transition we have observed here. The tricritical point is also present in the $(\infty,4)$ lattice, which is nothing but the Bethe lattice. This suggests that the suppression of the loop-back effect in the hyperbolic lattice is essential for the appearance of tricritical behavior.

Determination of the phase diagram in the region $\kappa > 1/4$ is challenging because the ground-state spin configuration becomes nontrivial, as has been discussed in Sec. II. Because the (5,4) hyperbolic lattice consists of pentagons, the lattice does not decouple into sublattices even when $J_1=0$. For the study of this region, we have to modify the CTMRG algorithm in order to treat ordered states with nontrivial spin patterns.

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